

# Stability of laminar Couette flow of compressible fluids

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**ABSTRACT** Cylindrical Couette flow is a subject where the main focus has long been on the onset of turbulence or, more precisely, the limit of stability of the simplest laminar flow. The theoretical framework of this paper is a recently developed action principle for hydrodynamics. It incorporates Euler-Lagrange equations that are in essential agreement with the Navier-Stokes equation, but applicable to the general case of a compressible fluid. The variational principle incorporates the equation of continuity, a canonical structure and a conserved Hamiltonian. The density is compressible, characterized by a general (non-polar) equation of state, and homogeneous.

The onset of instability is often accompanied by bubble formation. It is proposed that the limit of stability of laminar Couette flow may some times be related to cavitation. In contrast to traditional stability theory we are not looking for mathematical instabilities of a system of differential equations, but instead for the possibility that the system is driven to a metastable or unstable configuration. The application of this idea to cylindrical Couette flow reported here turns out to account rather well for the observations.

The failure of a famous criterion due to Rayleigh is well known. It is here shown that it may be due to the use of methods that are appropriate only in the case that the equations of motion are derived from an action principle.

## 1. Introduction

It is a common experience that action principles relate to strong predictive power. The present paper is motivated by the observation that a lack of a natural and successful approach to stability theory of compressible fluids is due to the descriptive nature of the standard theory, a theory that easily describes what is being observed but often without making actual predictions. In addition, it is not uncommon to make essential use of concepts that are natural only in the setting of an action principle. The most important instance is the energy concept; the stress tensor is another construct that is fully coherent only within an action principle. For this and several other reasons an action principle for hydrodynamics has been sought for more than a century; only now do we feel that one of the appropriate level of generality is available. (Fronsdal 1,2,3) It can be viewed as a minimal extension of the theory of potential flows and for this reason it is expected to have wide application to systems of moderate complexity. An application to electromagnetic theory of materials has been attempted (Fronsdal 4).

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In experiments initiated by Couette (5,6,7), and Mallock (8,9) a fluid was contained in the space between two concentric cylinders that could be made to



from the use of differential equations without the structural unity of an action principle and the application of particle concepts to fluids.

## 2. The action principle

The venerable action principle of Fetter and Walecka (13) hearks back to Bernoulli. It is extremely powerful wherever it can be used; that is, in the case of irrotational flows. It is especially efficient in the treatment of mixtures. In the case of cylindrical Couette flow, where more general flows play an important role, the favored approach has been that of the Navier-Stokes equation, as in Navier (14,15), Stokes (16), Taylor (10) and Jones (17). This equation has the great merit of predicting the existence of just two very special types of stationary flow, of which one is irrotational and the other can be characterized as ‘solid-body’ type. The existence of two types of flow is a very general and a very striking feature of hydrodynamics, first discussed by Couette (6). Later it turned out to be a dominant feature of superfluid Helium see Landau (18), Hall and Vinen (19), Feynman (20).

The action principle used in this paper is based on the following action for unary hydrodynamics,

$$A = \int d^3x dt \mathcal{L},$$

with the Lagrangian density

$$\mathcal{L} = \rho(\dot{\Phi} + \vec{X}^2/2 + \kappa \vec{X} \cdot \vec{\nabla} \Phi - \vec{\nabla} \Phi^2/2) - W[\rho]. \quad (2.1)$$

The variables are the density  $\rho$  and its canonical conjugate, the scalar velocity potential  $\Phi$  ( $\vec{v} = -\vec{\nabla} \Phi$ ), and the vector potential  $\vec{X}$ . These are precisely the variables that appear in a recent review by Fetter (21) on trapped Bose-Einstein condensates, except that in that paper the velocity  $\vec{X}$  is not a field variable but a fixed parameter that is interpreted as the velocity of a solid-body rotation. Compare also Hall and Vinen (19), a much cited paper that uses exactly the same variables. The scalar velocity potential is needed to generate the equation of continuity, essential to any application of hydrodynamics, but it does not furnish the expected number of four independent field variables of classical hydrodynamics. Two more are needed to accomodate rotational flows. An unconstrained vector field would go too far, by introducing too many independent variables, but this is remedied by the constraint

$$\vec{\nabla} \wedge \vec{m} = 0, \quad \vec{m} := \rho \vec{w}, \quad (2.2)$$

where the ‘vortex vector field’  $\vec{w}$  is

$$\vec{w} := \vec{X} + \kappa \vec{\nabla} \Phi. \quad (2.3)$$

The relevance of this constraint for the vortex flow was suggested by Lund and Regge (22).

The vector density  $\vec{m}$  is the momentum canonically conjugate to the field  $\vec{X}$ , identified by Lund and Regge as the ‘momentum’ of ‘non-equilibrium thermodynamics’. That important paper may have been the first to distinguish between the momentum  $\rho\vec{v}$  and the flow vector  $\rho\vec{v}$ . Imposing the constraint and fixing the gauge reduces the number of physical degrees of freedom carried by the field  $\vec{X}$  from 3 to 1, leaving only one additional pair of physical, canonically conjugate variables, besides the pair  $\rho, \Phi$ . The total number of field variables is 4, as in classical hydrodynamics.

The Euler Lagrange equations include the equation of continuity (variation of  $\Phi$ ),

$$\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v}) = 0, \quad \vec{v} := \kappa\dot{\vec{X}} - \vec{\nabla}\Phi,$$

the constraint (2.2) (variation of the gauge field), the wave equation

$$\frac{d}{dt}\vec{m} = 0 \tag{2.4}$$

(variation of  $\vec{X}$ ) and a generalized Bernoulli equation

$$\dot{\Phi} + \dot{\vec{X}}^2/2 + \kappa\dot{\vec{X}} \cdot \vec{\nabla}\Phi - \vec{\nabla}\Phi^2/2 = \mu + \text{constant} \tag{2.5}$$

(variation of the density). The functional  $W[\rho]$  is the internal energy density (for fixed entropy) and the functional

$$\mu[\rho] = \frac{\partial W}{\partial \rho}$$

is the chemical potential.

The physical meaning of the parameter  $\kappa$  varies according to the application and it will be clarified as different types of applications are studied.

The equations of motion all express conservation laws; in particular, (2.4) is the law of conservation of ‘momentum’. In the presence of any type of dissipation one or more of the conservation laws is violated; the usual assumption is that viscosity leads to violation of momentum conservation, (2.4) being replaced by

$$\frac{d}{dt}\vec{m} = \nu\rho\Delta\vec{v}, \tag{2.6}$$

where  $\nu$  is the kinematical viscosity; it may some times be taken to be a fixed, constant parameter. In the present context this equation is a generalisation of the Navier-Stokes equation. In particular, it agrees with it in requiring that any stationary flow must be harmonic.

In the case of a purely potential flow, when  $\dot{\vec{X}} = 0$  and the Fetter-Walecka theory applies, the imposition of a hydrodynamic equation of state, a relation between the density  $\rho$  and the pressure  $p$ , is enough to render the system determinant, since Eq.(2.5) relates the density to the velocity. For example, in the case of a polytropic fluid, when  $p \propto \rho^\gamma$  with the constant of proportionality determined by the specific entropy, the chemical potential  $\mu$  takes the form

$\rho^{\gamma-1} = \rho^{1/n}$ . In the present context it is usual to deal exclusively with incompressible liquids (the limit  $n \rightarrow \infty$ ), but we shall not restrict our treatment to that idealized situation. The density is compressible and the equation of state remains arbitrary. The specific motion known as laminar Couette flow is possible for a class of fluids, the general characteristics of which has not been elucidated.

In the general case ( $\dot{\vec{X}} \neq 0$ ) Eq.(2.5) is still sufficient to determine the density profile when both vector fields are specified. By (2.6), when  $\nu \neq 0$ , strictly stationary flow is possible only in the case that the vector field  $\vec{v}$  is harmonic,

$$\Delta \vec{v} = 0.$$

Consequently, in the presence of viscosity ( $\nu \neq 0$ ) any stationary motion that respects the symmetry of the cylinder is the sum of two types, the locally irrotational flow

$$-\vec{\nabla}\Phi = \frac{a}{r^2}(-y, x, 0)$$

and the ‘solid body’ type of flow

$$\dot{\vec{X}} = b(-y, x, 0),$$

with constants  $a, b$ .

This is the type of flow that is observed in the laboratory. The outer cylinder is rotated at a fixed angular velocity until a stationary state is obtained. Then the inner cylinder is rotated at a very slowly increasing rate until, at a certain critical angular velocity, the flow changes character. The classical problem is to predict the onset of this phenomenon in terms of the angular speeds  $\omega_1, \omega_2$  of the two cylinders.

The constraint (2.4) actually fixes the density profile of any harmonic flow, up to a multiplicative constant,

$$1/\rho \propto br^2 - \kappa a. \quad (2.7)$$

Since the fluid is compressible, the Stokes’ stream function will not appear in this work.

It is well known that the onset of turbulence is accompanied by bubble formation and some times with cavitation. Cavitation results when the motion overcomes the considerable tensile strength of the fluid, as when it is subjected to a strong sound wave; it is likely to depend almost entirely on the local density. If there is a critical density, of whatever type, then this implies a certain well defined critical value for the right hand side of Eq.(2.5) and if the relation between this function and the density is one-to-one, then the character of the flow is likely to change whenever this value of the chemical potential is reached.

**Suggested criterion for the stability  
of stationary, horizontal flow.**

Local breakdown of laminar flow is expected to occur if, when and where the chemical potential on the right side of (2.5) reaches a certain critical value  $C$ , which implies the relation

$$\dot{\vec{X}}^2/2 + \kappa \dot{\vec{X}} \cdot \vec{\nabla} \Phi - \vec{\nabla} \Phi^2/2 = C. \quad (2.8)$$

The quantity on the left depends on the choice of boundary conditions and on the selected point in the fluid. The value of the constant  $C$  is the only adjustable parameter.

Supposing that the motion is horizontal and circular; as the angular speed of the inner cylinder is increased, and the number  $C$  consequently is decreased, cavitation or a similar local disturbance is expected to occur, at first, on some particular points of the fluid. Let  $\omega_1, \omega_2$  be the angular speeds of the inner, resp. outer cylinder. For any particular value of  $C$ , and at any fixed point in the fluid, this will define a locus of points in the  $\omega_1, \omega_2$  plane. The criterion will predict the onset of the phenomenon at each point and the shapes of these loci. It is well known that, for a particular set of boundary conditions, cavitation often appears on smooth submanifolds in the fluid. The criterion will predict the location of such points and the shape of these manifolds.

### 3. Calculation of critical boundary

For horizontal, laminar flow the velocities are

$$-\vec{\nabla} \Phi = \frac{a}{r^2}(-y, x, 0) = \frac{a}{r} \hat{\theta}, \quad \dot{\vec{X}} = b(-y, x, 0) = br \hat{\theta}, \quad (3.1)$$

and the manifold of instability in  $(a, b)$  is given by

$$b^2 r^2/2 + \kappa ab - a^2/2r^2 = C_{\text{cr}}, \quad r_1 < r < r_2. \quad (3.2)$$

The mass flow velocity is  $\vec{v} = \kappa \dot{\vec{X}} - \vec{\nabla} \Phi$ . In terms of the boundary conditions, the angular velocities  $(\omega_1, \omega_2)$  and the cylinder radii  $r_1, r_2$  we have

$$a = r_1^2 r_2^2 \frac{\omega_1 - \omega_2}{r_2^2 - r_1^2}, \quad \kappa b = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}.$$

For any value of the radial coordinate  $r$ , the locus (3.2) in the  $a, b$  plane is a hyperbola with asymptotes found by replacing  $C$  by 0,

$$r^2 b/a = -\kappa \pm \sqrt{1 + \kappa^2}.$$

The boundary conditions translate this to the asymptotes in the  $\omega_1, \omega_2$  plane,

$$\frac{\omega_1}{\omega_2} = \frac{\frac{r^2}{r_1^2} + k_{\pm}}{\frac{r^2}{r_2^2} + k_{\pm}}, \quad k_{\pm} = -\kappa^2 \pm \kappa \sqrt{1 + \kappa^2}. \quad (3.3)$$

The observations by Andreck *et al* (12) are shown in Fig.2, where the asymptotes are in the first and fourth quadrant of the  $(\omega_1, \omega_2)$  plane. This feature need not be true for other fluids, but we wish to know if it can be reproduced by the model. It requires that

$$\frac{r^2}{r_1^2} > \kappa^2 + |\kappa\sqrt{1+\kappa^2}| > \frac{r^2}{r_2^2}. \quad (3.4)$$

It is the region shown colored or shaded in Fig 3.

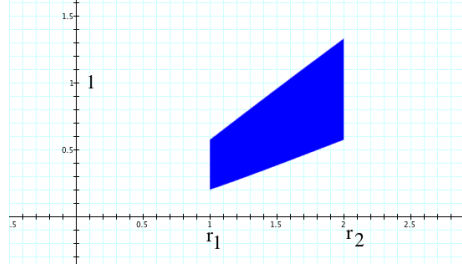


Fig.3. The abscissa is the radial coordinate and the ordinate is the value of the parameter  $\kappa$ . The inequalities (3.4) are satisfied in the shaded regions.

If cavitation occurs first at the inner surface we have by (3.4)

$$\frac{\omega_1}{\omega_2} = \frac{1 + k_{\pm}}{\frac{r_2^2}{r_1^2} + k_{\pm}}, \quad k_{\pm} = -\kappa^2 \pm \kappa\sqrt{1+\kappa^2}. \quad (3.5)$$

See Fig 4. There is one pair of predicted slopes for each value of  $\kappa$ .

### Result

In the figures, the ordinate represents the value of the parameter  $\kappa$ . Any value of  $\kappa$  gives two slopes. The positive value is not adjustable, the prediction is 1.2, a bullseye for the theory. The negative asymptote varies from the negative  $\omega_2$  axis to the positive  $\omega_1$  axis as  $\kappa$  runs a vary narrow range; the experimental value of -.73 of the slope indicates that  $\kappa = .541$ , a highly precise determination of this parameter.

In the first figure we are looking at the inner boundary,  $r = r_1$ . If instead cavitation occurs first at the outer surface we have

$$\frac{\omega_1}{\omega_2} = \frac{\frac{r_2^2}{r_1^2} + k_{\pm}}{1 + k_{\pm}}, \quad k_{\pm} = -\kappa^2 \pm \kappa\sqrt{1+\kappa^2}. \quad (3.6)$$

Fig. 4B represents the same information as Fig 4A, relative to the outer boundary. Data points between the singular curves are interpolated smoothly and uniformly. For any value of  $\kappa$ , the bubbles appear first at the inner surface as the slope  $|\omega_1/\omega_2|$  is increased. This too is believed to be in accord with experience.

We used  $r_1/r_2 = .883$  as in the Andereck experiment. The negative asymptote (in the NW quadrant) varies (with  $\kappa$ ) from the negative  $\omega_2$  axis to the positive  $\omega_1$  axis. The value  $\kappa \approx .541$  amounts to an experimental determination of this parameter for water at normal conditions.

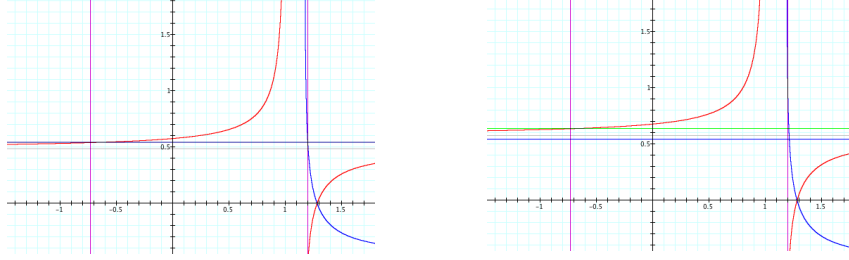


Fig 4A,B. The relations (3.5). The red (purple) go with the plus (minus) sign. The ordinate is the value of  $\kappa$ , the abscissa the asymptotic slope  $\omega_1/\omega_2$ ; this figure applies at the inner cylinder. The measured negative slope is -.73. Eq.(3.5) gives  $\kappa = .541$ . For the same value of  $\kappa$  the positive slope is 1.2. Fig.4B as Fig.4A, but at the outer boundary.

#### 4. Rayleigh's criterion

The most venerable result concerning stability of Couette flow is due to Rayleigh (23), (24). Applied to the flow  $(a/r^2 + b)(-y, x, 0)$  it states that this flow is stable only if

$$b/a > (r_2/r_1)^2.$$

According to this result, Couette flow between cylinders rotating in opposite directions would always be unstable. See for example Drazin and Reid (25), page 79, Chandrasekhar (26), Koschmieder (27). This is at variance with observations and it is of some interest to ask what is wrong with the proof.

Rayleigh's method employs a mixture of ideas from fluid dynamics and particle physics; it relies, in particular, on "conservation of angular momentum". But angular momentum conservation is a concept that, like energy conservation, belongs to action principles. To illustrate the meaning of angular momentum conservation for Couette flow we may consider for simplicity, the special case of irrotational flow, when the vortex field is absent and  $\vec{v} = -\vec{\nabla}\Phi$ . In this case the Navier-Stokes equation and the equation of continuity are the Euler-Lagrange equations of the action

$$A = \int d^3x \left( \rho(\dot{\Phi} - \vec{v}^2/2) - f(\rho, T) - sT \right), \quad \vec{v} := -\vec{\nabla}\Phi,$$

where  $f$  is the free energy density and  $s$  is the entropy density or, when as usual the specific entropy  $S = s/\rho$  is constant. When the temperature is eliminated



with the help of the adiabatic condition, one obtains

$$A = \int dt \int d^3x \left( (\rho(\dot{\Phi} - \vec{v}^2/2) - W[\rho]) \right), \quad (4.1)$$

where  $W[\rho]$  is the internal energy density for a fixed value of the specific entropy density. The action of an infinitesimal rotation on the fields  $\rho$  and  $\Phi$  is generated by the operators

$$G_i = \epsilon_{ijk} x_j \partial_k, \quad \text{or} \quad G = \vec{\alpha} \cdot \vec{x} \wedge \vec{\nabla}, \quad (4.2)$$

with  $\vec{\alpha}$  a constant. The effect on the action is

$$\delta A = \int dt \int d^3x G \mathcal{L}.$$

With the help of the equations of motion one obtains the “integrated part”

$$\int dt \int d^3x G \mathcal{L} = \int dt \int d^3x \left( \frac{d}{dt} (\rho G \Phi) + \vec{\nabla} \cdot ((\rho G \Phi) \vec{v}) \right)$$

or

$$\frac{d}{dt} \int_{\Sigma} d^3x \rho G \Phi - \int_{\Sigma} d^3x G p = - \int_{\partial \Sigma} (\rho G \Phi) \vec{v} \cdot \vec{d}\sigma. \quad (4.3)$$

In the second term  $Gp$  is the torque density; we have replaced the Lagrangian density  $\mathcal{L}$  by its on shell value, the thermodynamic pressure  $p$ . The density of angular momentum is thus

$$\rho \vec{x} \wedge \vec{\nabla} \dot{\Phi} = -\rho \vec{x} \wedge \vec{v}.$$

As is the case with other local conservation laws, conservation means that any increase in angular momentum within  $\Sigma$  is accounted for by the inflow through the boundary ... except that in this case the action for the fluid is not invariant and there is the second term on the left side of eq.(4.3), the torque.

Rayleigh assumed that if an element of fluid executes a virtual displacement from one position to another, then the quantity  $\int d^3x \rho \vec{x} \wedge \vec{v}$  would remain constant. If the volume element moves with the fluid, then the velocity is normal to the surface element and the surface integral on the right hand side of (4.3) vanishes; the integral of the angular momentum density over  $\Sigma$ , would indeed have been constant, were it not for the torque term. This term,  $-Gp$ , is analogous to the term  $-(d/dt)\mathcal{L}$  in the definition of the Hamiltonian density. The fact that the latter is a time derivative (a boundary term in the time direction) does not mean that it can be ignored. The torque term is equally relevant, although it can be expressed as a surface integral. It represents the torque applied to the volume element by the surrounding fluid.

Compare the expression for energy conservation

$$\frac{d}{dt} \int d^3x \left( \dot{\rho} \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} + \dot{\Phi} \frac{\partial \mathcal{L}}{\partial \dot{\rho}} - \mathcal{L} \right) = \int_{\partial \Sigma} (\rho \dot{\Phi}) \vec{v} \cdot \vec{d}\sigma = \int_{\partial \Sigma} (h + p) \vec{v} \cdot \vec{d}\sigma \quad (4.4)$$

The torque term  $-Gp$  in (4.3) corresponds to the term  $-(d/dt)\mathcal{L}$  in (4.4) and like that term it cannot be ignored. The scalar factor  $h + p$  in the energy flux density is the enthalpy density.

The local expression for the conservation law (4.3) is

$$\frac{d}{dt}(\rho \vec{x} \wedge \vec{v}) + \vec{x} \wedge \vec{\nabla} p - \vec{\nabla} \cdot \rho G \Phi \vec{v} = 0,$$

or

$$\vec{x} \wedge \left( \frac{d}{dt} \rho \vec{v} + \sum_i \partial_i \rho \vec{v} v_i + \vec{\nabla} p \right) = 0.$$

Making use of the equation of continuity we simplify this to

$$\vec{x} \wedge \left( \rho \dot{\vec{v}} + \rho (\vec{v} \cdot \vec{\nabla}) \vec{v} + \vec{\nabla} p \right) = 0.$$

This is essentially the Navier-Stokes equation. (Precisely the Navier equation is obtained when we consider the conservation of linear momentum instead of angular momentum.)

A principle feature of action principles is minimization of the energy. There is a functional that, for any physical configuration, is stationary with respect to a specified set of variations of the dynamical variables. In this case what is needed is a determination of the sign of second order variations. This method can not be applied in Navier-Stokes theory, in general, because there is no unperturbed configuration that is a minimum in any sense. But it can be used in the special case of Fetter-Walecka theory, and with the more general Lagrangian (2.1). The classical method of linear analysis is to calculate infinitesimal, harmonic variations, each variable taking the form.

$$\psi = \psi_0 + d\psi, \quad d\psi \propto e^{i(\vec{k} \cdot \vec{x} + \omega t)},$$

with  $\vec{k}, \omega$  constant, possibly complex. In the case of an action principle these modes extremize the total energy and there remains the task of distinguishing minima (stable modes) from maxima (unstable modes). What is frequently overlooked is the need to verify that the modes are located at an ‘extremum’, something that is clear only in the case when the equations of motion are the Euler-Lagrange equations of an action principle.

In the general case of a homogeneous fluid flow linear analysis of the action principle leads to two types of sound propagation (with  $\vec{k}, \omega$  real).

The instability criterion proposed in section 2 is of a different order. On the assumption that the equations of motion have regular solutions, stable to all orders, any breakdown would be associated with the fact that the dynamical variables are taking values outside the domain of validity of the equations of motion. This may be the case if the predicted density turns negative in a neighborhood. Before that happens the pressure may turn negative, the configuration becomes metastable and decays quickly because of irregularities in the flow. What is likely to happen in that case is the onset of cavitation on a small

or on a larger scale, which would imply a behavior distinct from that predicted by the equations of motion, until another stable flow is discovered. We have proposed that the observed onset of Taylor flow may have this interpretation and we have found that this results in a prediction that agrees quite well with the observations.

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